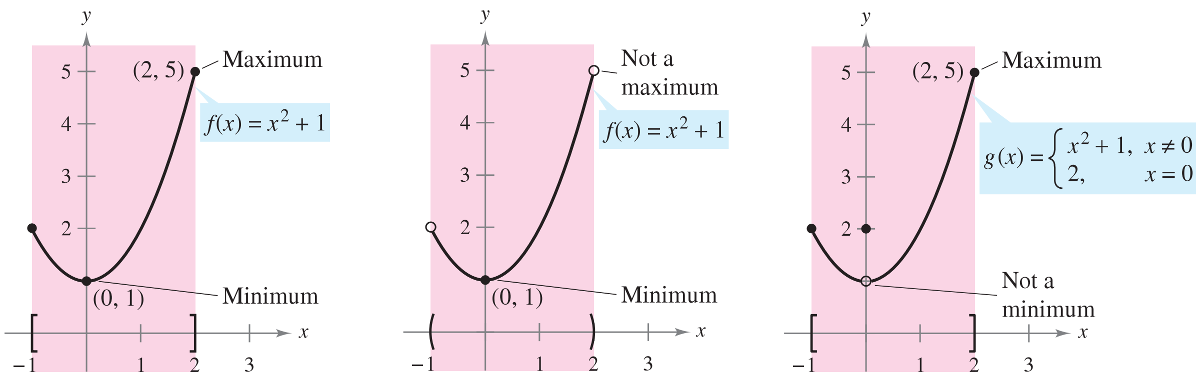
Math 150, ***Lecture Notes- Bonds*** **Name____________**

Section 3.1 *Extrema on an Interval*

Extrema of a Function

In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? Where is the function increasing? Where is it decreasing? In this chapter you will learn how derivatives can be used to answer these questions. You will also see why these questions are important in real-life applications.



(a) f is continuous, $[-1, 2]$ is closed. **(b)** f is continuous, $(-1, 2)$ is open. **(c)** g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum** on the interval.

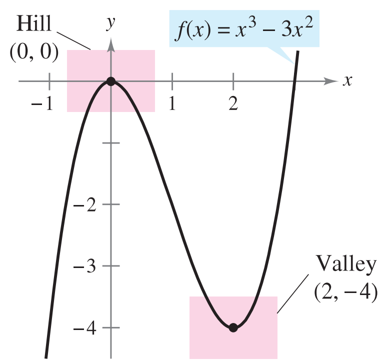
A function need not have a minimum or a maximum on an interval.

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Relative Extrema and Critical Numbers

In Figure 3.2, the graph of $f(x) = x^3 - 3x^2$ has a **relative maximum** at the point $(0, 0)$ and a **relative minimum** at the point $(2, -4)$. Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph. Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).



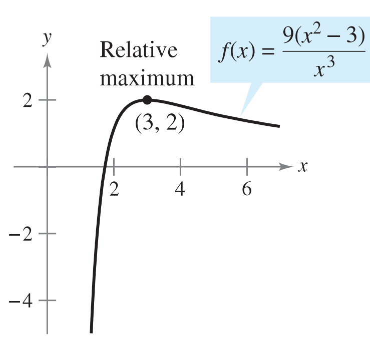
f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.
Figure 3.2

Definition of Relative Extrema

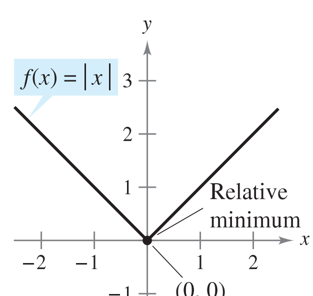
1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima.

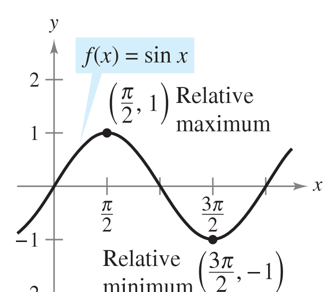
Ex.1 Find the value of the Derivative at Relative Extrema



(a) $f'(3) = 0$



(b) $f'(0)$ does not exist.

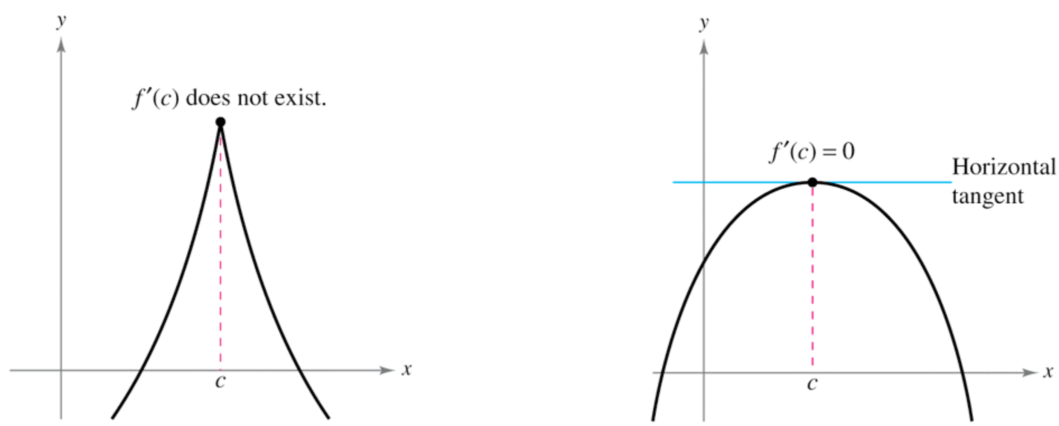


(c) $f'\left(\frac{\pi}{2}\right) = 0;\ f'\left(\frac{3\pi}{2}\right) = 0$

Note in Example 1 that at each relative extremum, the derivative either is zero or does not exist. The x-values at these special points are called **critical numbers.** Figure 3.4 illustrates the two types of critical numbers. Notice in the definition that the critical number c has to be in the domain of f, but c does not have to be in the domain of f'.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Finding Extrema on a Closed Interval

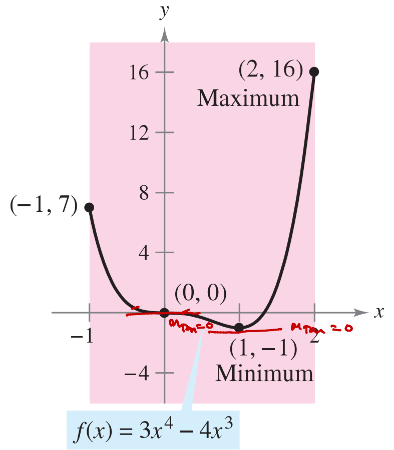
Theorem 3.2 states that the relative extrema of a function can occur *only* at the critical numbers of the function. Knowing this, you can use the following guidelines to find extrema on a closed interval.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Ex.2 Use calculus to find the extrema of $f(x)=3x^4-4x^3$ on the interval $[-1,2]$.



On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

Figure 3.5

$f'(x) = 12x^3 - 12x^2$

critical numbers: 0 & 1

(A) $f'(x) = 0$ or (B) $f'(x)$ is undefined

$12x^3 - 12x^2 = 0$

$f'(x) = 12x^3 - 12x^2$

Never

$12x^2(x-1) = 0$

either

$12x^2 = 0$, or $x - 1 = 0$

$x^2 = 0$ $\quad x = 1$

$x = 0$

check

From critical numbers:

$f(0) = 3(0)^4 - 4(0)^3 = 0$

$f(1) = 3(1)^4 - 4(1)^3 = 3 - 4 = -1$ $\boxed{\text{Low}}$ Abs. Min

From endpoints:

$f(-1) = 3(-1)^4 - 4(-1)^3 = 3 + 4 = 7$

$f(2) = 3(2)^4 - 4(2)^3 = 3\cdot16 - 4\cdot8$

$= 48 - 32$

$= 16$ $\boxed{\text{High}}$

Abs. MAX

In Figure 3.5, note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema.*

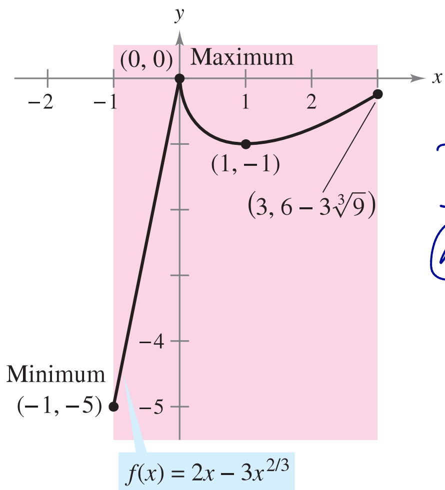
ON A TEST

Ex.3 Use calculus to find the extrema of $f(x) = 2x - 3x^{\frac{2}{3}}$ on the interval $[-1, 3]$.

$f'(x) = 2 - 3 \cdot \frac{2}{3} x^{-1/3}$

$f'(x) = 2 - 2x^{-1/3}$

critical Numbers: 0 & 1



On the closed interval $[-1, 3]$, f has a minimum at $(-1, 5)$ and a maximum at $(0, 0)$.

(A) $f'(x) = 0$

$2 - 2x^{-1/3} = 0$

$2 = 2x^{-1/3}$

$2 = \frac{2}{x^{1/3}}$

$x^{1/3} = \frac{2}{2}$

$x^{1/3} = 1$

$(x^{1/3})^3 = (1)^3$

$x = 1$

(B) $f'(x)$ is undefined

$2 - 2x^{-1/3}$

$= 2 - \frac{2}{x^{1/3}}$

$x^{1/3} = 0$

$(x^{1/3})^3 = (0)^3$

$x = 0$

From critical Numbers

$f(0) = 2(0) - 3(0)^{2/3}$

$= 0$ ✓ Highest

$f(1) = 2(1) - 3(1)^{2/3}$

$= 2 - 3 \cdot 1$

$= -1$ ✓

end points

$f(-1) = 2(-1) - 3(-1)^{2/3}$

$= -2 - 3(1)$

$= -5$ ✓ LOWEST

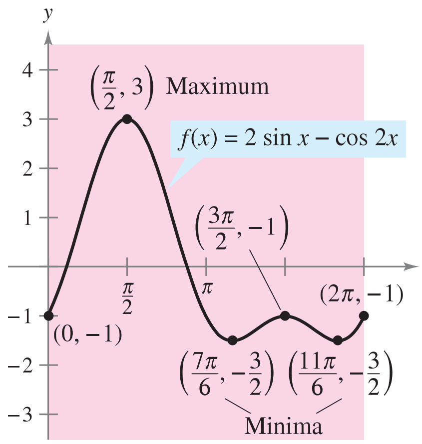
$f(3) = 2(3) - 3(3)^{2/3}$

$= 6 - 3\sqrt[3]{9}$ ✓

Abs. max at (0, 0)

Abs. min at (-1, -5)

Ex.4 Use calculus to find the extrema of $f(x)=2\sin(x)-\cos(2x)$ on the interval $[0,2\pi]$.



On the closed interval $[0, 2\pi]$, f has two minima at $(7\pi/6, -3/2)$ and $(11\pi/6, -3/2)$ and a maximum at $(\pi/2, 3)$.

$f'(x) = 2\cos(x) - [-\sin(2x)]\cdot\underbrace{(2)}_{\text{chain Rule}}$

$\boxed{f'(x) = 2\cos(x) + 2\sin(2x)}$

critical numbers:

Ⓐ $f'(x) = 0$, Ⓑ $f'(x)$ is undefined

Never

$0 = 2\cos(x) + 2\sin(2x)$

$0 = 2\cos(x) + 2[2\sin(x)\cos(x)]$

$0 = 2\cos(x) + 4\sin(x)\cos(x)$

$0 = 2\cos(x)[1 + 2\sin(x)]$

Either

$0 = 2\cos(x)$, or $0 = 1 + 2\sin(x)$

$0 = \cos(x)$ | $-1 = 2\sin(x)$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ | $-\frac{1}{2} = \sin(x)$

$x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$

from critical Numbers

$f\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right) - \cos\left(2\left(\frac{\pi}{2}\right)\right)$
$= 2\cdot 1 - \cos(\pi)$
$= 2 - (-1) = 2 + 1 = 3$

$f\left(\frac{3\pi}{2}\right) = 2\sin\left(\frac{3\pi}{2}\right) - \cos\left(2\left(\frac{3\pi}{2}\right)\right)$
$= 2(-1) - \cos(3\pi)$
$= -2 - (-1) = -2 + 1 = -1$

$f\left(\frac{7\pi}{6}\right) = 2\sin\left(\frac{7\pi}{6}\right) - \cos\left(2\cdot\frac{7\pi}{6}\right)$
$= 2\left(-\frac{1}{2}\right) - \cos\left(\frac{7\pi}{3}\right)$
$= -1 - \frac{1}{2} = -\frac{3}{2}$

$f\left(\frac{11\pi}{6}\right) = 2\sin\left(\frac{11\pi}{6}\right) - \cos\left(2\left(\frac{11\pi}{6}\right)\right)$
$= 2\left(-\frac{1}{2}\right) - \cos\left(\frac{11\pi}{3}\right)$
$= -1 - \left(\frac{1}{2}\right) = -1 - \frac{1}{2} = -\frac{3}{2}$

ABS MAX at $\left(\frac{\pi}{2}, 3\right)$

ABS min at $\left(\frac{7\pi}{6}, -\frac{3}{2}\right)$ & $\left(\frac{11\pi}{6}, -\frac{3}{2}\right)$

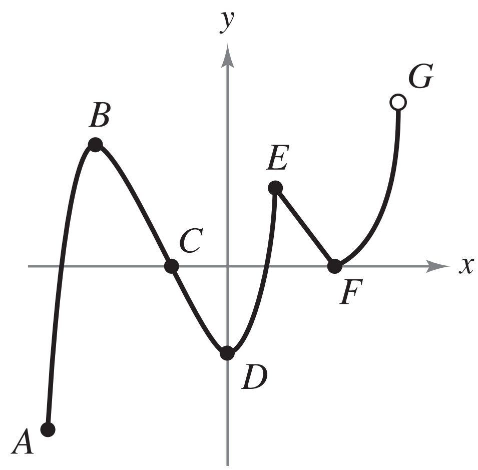
From ENDPOINTS

$f(0) = 2\sin(0) - \cos(2\cdot 0)$
$= 2\cdot 0 - \cos(0)$
$= 0 - 1 = -1$

$f(2\pi) = 2\sin(2\pi) - \cos(2\cdot 2\pi)$
$= 2\cdot 0 - \cos(4\pi)$
$= 0 - 1$
$= -1$

On a Test

Ex.5 Decide whether each labeled point is an absolute maximum or minimum, a relative maximum or minimum, or neither.



A – Absolute Minimum

B – Relative Maximum

C – neither

D – Relative Minimum

E – Relative Maximum

F – Relative Minimum

G – neither